Cauchy-Goursat theorem

Monday, December 7, 2020 9:06 AM

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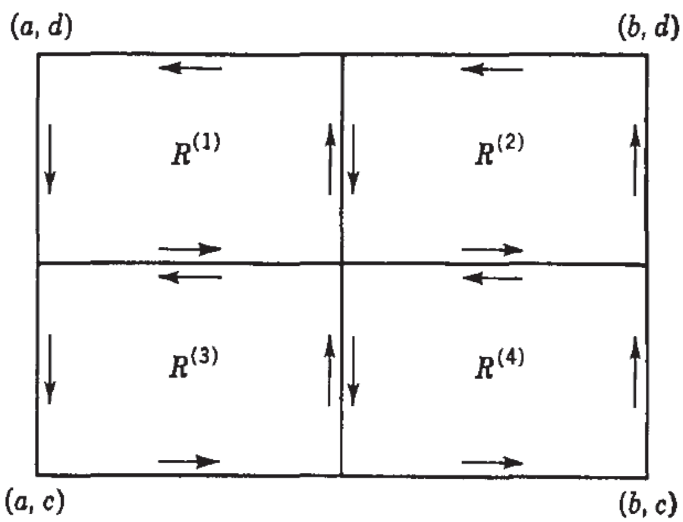
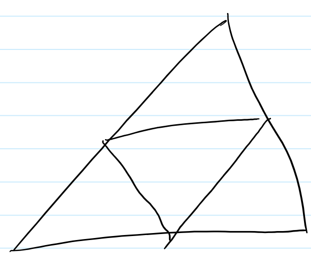
Theorem (Cauchy-Goursat).
Let R be a rectangle, $f \in \mathcal{A}(R)$.
Then $\oint_{\partial R} f(z)\,dz = 0$.

Corollary (Cauchy). Let $f \in \mathcal{A}(B(z_0, r))$.
Then for any closed $\gamma \subset B(z_0, r)$,
$\oint_\gamma f(z)\,dz = 0$. There exists $F \in \mathcal{A}(B(z_0, r))$:
$F'(z) = f(z)$ $\forall z \in B(z_0, r)$.

Proof of Theorem.





Key idea: cut R into four equal rectangles: $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$. Observe:

$$\oint_{\partial R} f(z)\,dz = \sum_{j=1}^{4} \oint_{\partial R^{(j)}} f(z)\,dz$$

$$\ell(\partial R^{(j)}) = \frac{1}{2}\ell(\partial R). \quad \operatorname{diam}(R^{(j)}) = \frac{1}{2}\operatorname{diam} R.$$

Now: assume $\left|\oint_{\partial R} f(z)\,dz\right| = A \neq 0$. Let $R_0 := R$

Then $\exists\, R^{(j)}$: $\left|\oint_{\partial R^{(j)}} f(z)\,dz\right| \geq \frac{A}{4}$. Take $R_1 = R^{(j)}$ for such j.

Repeat: $\exists\, R_2$ - subrectangle of R_1: $\left|\oint_{\partial R_2} f(z)\,dz\right| \geq \frac{1}{4}\left|\oint_{\partial R_1} f(z)\,dz\right| \geq 4^{-2}A.$

R_n - subrectangle of R_{n-1}:

$$\left|\oint_{\partial R_n} f(z)\,dz\right| \geq A\cdot 4^{-n}.$$

$$\ell(\partial R_n) = 2^{-n}\,\ell(R)$$

$$\operatorname{diam}(R_n) = 2^{-n}\operatorname{diam} R.$$

Let $\{z^*\} = \bigcap R_n$ (non-empty, $\operatorname{diam} \bigcap R_n \leq 2^{-n}\operatorname{diam} R\ \forall n \Rightarrow \bigcap R_n$ is one point)

Pick $r>0$: $f \in \mathcal{A}(B(z^*,r))$ (exists, since $z^* \in R$, f is analytic in an open $U \supset R$).

$f'(z_*)$ exists. So $\forall \varepsilon>0\ \exists\, 0<\delta<r$: $|z-z^*|<\delta \Rightarrow \left|\frac{f(z)-f(z^*)}{z-z^*} - f'(z^*)\right| < \varepsilon.$

Or $|f(z)-f(z^*)-f'(z^*)(z-z^*)| < \varepsilon\,|z-z^*|.$

Take n: $2^{-n}\operatorname{diam}(R) < \delta$. Then $\partial R_n \subset B(z^*,\delta)$

$$\oint_{\partial R_n} f(z)\,dz = \overbrace{\oint_{\partial R_n}\big(f(z)-f(z^*)-f'(z^*)(z-z^*)\big)\,dz}^{\mathrm{I}} + \underbrace{\oint_{\partial R_n} f(z^*)\,dz}_{\mathrm{II}} + \underbrace{\oint_{\partial R_n} f'(z^*)(z-z^*)\,dz}_{\mathrm{III}}$$

$f(z^*) = (z f(z^*))'$

$f(z^*)(z-z^*) = \left(\frac{f(z^*)}{2}(z-z^*)^2\right)'$, so $\mathrm{II} = \mathrm{III} = 0.$

so $II = III = 0$.

$f(z^*)(z-z^*) = \left(\frac{f(z^*)}{2}(z-z^*)^2\right)'$

But $|f(z) - f(z^*) - f'(z^*)(z-z^*)| < \varepsilon |z-z^*| \leq \varepsilon \,\mathrm{diam}(R_n)$

so

$$4^{-n} A \leq \left|\oint_{\partial R_n} f(z)\,dz\right| = |I| \leq (\varepsilon\, \mathrm{diam}(R_n))(\ell(\partial R_n)) = 4^{-n}\varepsilon\, \mathrm{diam}(R)\, \ell(\partial R)$$

so $4^{-n} A \leq \varepsilon 4^{-n} \mathrm{diam}(R)\, \ell(\partial R) \quad \forall \varepsilon > 0$

$A \leq \varepsilon\, \mathrm{diam}(R)\, \ell(\partial R)$

Assumed $A > 0$. Take $\varepsilon = \dfrac{A}{2\,\mathrm{diam}(R)\,\ell(\partial R)}$

<u>Get</u>: $A \leq \frac{A}{2}$. – contradiction! $\blacksquare$

<u>Theorem</u> (Improved Goursat)

Let R be a rectangle, $R' := R \setminus \{z_1, \dots, z_n\}$ $(z_j \in \mathrm{Int}\, R)$. Let $f \in A(R')$, and

$\lim_{z \to z_j} (z - z_j) f(z) = 0 \quad \forall j$.

Then $\int_{\partial R} f(z)\,dz = 0$.

$f(x) = g(x)$, $x \notin S$, $|S| = 0$

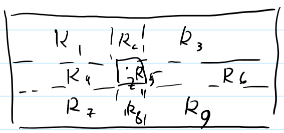
$\int f(x)\,dx = \int g(x)\,dx$

Remark. If f is continuous at z_i, or even locally-bounded ($\exists M, \delta: 0<|z-z_i|<\delta \Rightarrow |f(z)|<M$), then automatically $\lim\limits_{z\to z_i}(z-z_i)f(z)=0$.

Corollary. Let $f\in A(B(z_0,r)\setminus\{z_1,\dots,z_n\})$. Let $\forall i: \lim\limits_{z\to z_i}(z-z_i)f(z)=0$. Then for any closed $\gamma\subset B(z_0,r)\setminus\{z_1,\dots,z_n\}$ $\oint_\gamma f(z)\,dz=0$. There exists $F\in A(B(z_0,r)\setminus\{z_1,\dots,z_n\})$: $F'(z)=f(z)$ $\forall z\in B(z_0,r)$.

Proof of Improved Goursat.

By subdivision, can assume that $n=1$.



Cut R into 9 rectangles, as shown. For $j\neq 5$, $f\in A(R_j)$, so $\oint_{\partial R_j} f(z)\,dz=0$.

Observe:
$$\oint_{\partial R} f(z)\,dz=\sum_{j=1}^{9}\oint_{\partial R_j} f(z)\,dz=\oint_{\partial R_5} f(z)\,dz.$$

Fix $\varepsilon>0$. Choose $\delta>0$: $|z-z_1|<\delta \Rightarrow |f(z)(z-z_1)|<\varepsilon \Rightarrow |f(z)|<\frac{\varepsilon}{|z-z_1|}$.

Take R_5 to be a square of size $\frac{\delta}{2}$ centered at z_1.

Then $z\in\partial R_5 \Rightarrow \frac{\delta}{4}\le|z-z_1|\le\frac{\delta}{\sqrt{2}}<\delta$, so $z\in\partial R_5\Rightarrow |f(z)|<\frac{\varepsilon}{|z-z_1|}$

So
$$\left|\oint_{\partial R} f(z)\,dz\right|=\left|\oint_{\partial R_5} f(z)\,dz\right|\le\varepsilon\int_{\partial R_5}\frac{|dz|}{|z-z_1|}\le\varepsilon\cdot\underset{|z-z_1|\ge\frac{\delta}{4}}{\frac{4}{\delta}}\cdot\overset{\ell(\partial R_5)}{\overset{\|}{2\delta}}=8\varepsilon.$$

Since ε is arbitrary, $\oint_{\partial R} f(z)\,dz=0$.